Modeling Debt
Solving For Debt IRR - Discrete-Time

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In order for a borrower to obtain a loan from a bank the borrower is required to obtain a loan guarantee from a third-party. The guarantor is obligated to step in and make the bank whole should the borrower default on the loan. The value of the guarantee should equal the present value of the credit spread over the term of the loan. In this white paper we will calculate that credit spread. To that end we will work through the following hypothetical problem from Part I...

Our Hypothetical Problem

We are given the following model parameters...

Table 1: Model Parameters

Symbol	Description	Value
D_0	Debt principal balance at time zero (\$)	100,000.00
P	Periodic debt payment amount (\$)	1,574.96
B	End-of-term balloon payment (\$)	25,000.00
G	Cost of the guarantee (paid by the borrower) (\$)	9,500.00
R	Annual contractual interest rate (%)	6.00
K	Annual risk-adjusted discount rate (%)	6.50
N	Number of annual time periods (#)	12
T	Total number of time periods (#)	60

Our task is to answer the following questions...

Question 1: What is the IRR on the debt excluding the guarantee.

Question 2: What is the IRR on the debt including the guarantee.

Question 3: What is the credit spread on the debt.

Debt Valuation Equations

We will define the variable κ to be the periodic risk-adjusted discount rate. Using the data in Table 1 above the equation for the periodic discount rate is...

$$\kappa = \left(1 + K\right)^{\frac{1}{N}} - 1 \tag{1}$$

We will define the variable θ to be the periodic discount factor, which is the discount factor applied to cash flow in each discrete time period. Using Equation (1) above and the data in Table 1 above the equation for the periodic discount factor is...

$$\theta = \frac{1}{1 + \kappa} \text{ ...where... } \frac{\delta \theta}{\delta \kappa} = -\left(1 + \kappa\right)^{-2} \tag{2}$$

We will define the variable V_0 to be the market value of debt at time zero. Using Equation (2) above and the data in Table 1 above the equation for debt value at time zero is...

$$V_0 = \sum_{t=1}^{T} P\, \theta^t + B\, \theta^T = P \sum_{t=1}^{T} \theta^t + B\, \theta^T \tag{3}$$

Using a polylogarithm of order zero the solution to Equation (3) above is... [1]

$$V_0 = P\,\frac{\theta\,(1-\theta^T)}{1-\theta} + B\,\theta^T \tag{4}$$

The derivative of Equation (4) above with respect to the variable θ is...

$$\frac{\delta V_0}{\delta\theta} = P\Bigg(1+\theta^T(\theta\,T - T - 1)\Bigg)\Bigg(\theta - 1\Bigg)^{-2} \tag{5}$$

Using Equations (2) and (5) above the derivative of Equation (4) above with respect to the variable κ is...

$$\frac{\delta V_0}{\delta\kappa} = \frac{\delta V_0}{\delta\theta}\,\frac{\delta\theta}{\delta\kappa} = -\Bigg[P\Bigg(1+\theta^T(\theta\,T - T - 1)\Bigg)\Bigg(\theta - 1\Bigg)^{-2} + T\,B\,\theta^{T-1}\Bigg]\Bigg(1+\kappa\Bigg)^{-2} \tag{6}$$

The Answers To Our Hypothetical Problem

Using Equation (1) above and the data in Table 1 above the equation for the periodic discount rate is...

$$\kappa = \Bigg(1 + 0.0650\Bigg)^{\frac{1}{12}} - 1 = 0.005262 \tag{7}$$

Using Equations (2) and (1) above the equaton for the periodic discount factor is...

$$\theta = \frac{1}{1 + 0.005262} = 0.994766 \tag{8}$$

Question 1: What is the IRR on the debt excluding the guarantee.

If we set the guess value of κ to Equation (7) above, and F(actual κ) = Debt principal at time zero, then the IRR on the debt via a Newton-Raphson methodology for solving non-linear equations is... [2]

Iteration	new κ	=	κ guess	F(κ)	F(κ guess)	dF(κ guess)
1	0.004998	=	0.005262	100,000.00	99,100.62	-3,410,134.15
2	0.005000	=	0.004998	100,000.00	100,007.07	-3,452,620.79
3	0.005000	=	0.005000	100,000.00	99,999.99	-3,452,288.75
4	0.005000	=	0.005000	100,000.00	100,000.00	-3,452,289.26
5	0.005000	=	0.005000	100,000.00	100,000.00	-3,452,289.26

Using the last entry in the table above, the annualized IRR on the debt excluding the guarantee is...

$$\text{IRR excluding guarantee} = \Bigg(1 + 0.005000\Bigg)^{12} - 1 = 6.17\% \tag{9}$$

Question 2: What is the IRR on the debt including the guarantee.

If we set the guess value of κ to Equation (7) above, and F(actual κ) = Debt principal at time zero minus the value of the guarantee, then the IRR on the debt via a Newton-Raphson methodology for solving non-linear equations is... [2]

Iteration	new κ	=	κ guess	F(κ)	F(κ guess)	dF(κ guess)
1	0.007784	=	0.005262	90,500.00	99,100.62	-3,410,134.15
2	0.007939	=	0.007784	90,500.00	90,970.42	-3,032,457.91
3	0.007939	=	0.007939	90,500.00	90,500.55	-3,010,827.70
4	0.007939	=	0.007939	90,500.00	90,500.00	-3,010,802.14
5	0.007939	=	0.007939	90,500.00	90,500.00	-3,010,802.20

Using the last entry in the table above, the annualized IRR on the debt excluding the guarantee is...

$$\text{IRR including guarantee} = \Bigg(1 + 0.007939\Bigg)^{12} - 1 = 9.95\% \tag{10}$$

Question 3: What is the credit spread on the debt.

Using Equations (9) and (10) above the credit spread (spread to cover credit losses) on the debt is...

$$\text{Credit spread} = 9.95\% - 6.17\% = 3.79\% \tag{11}$$

References

[1] Gary Schurman, *Polylogarithms Of Order Zero,* May, 2019.

[2] Gary Schurman, *The Newton Raphson Method For Solving Non-Linear Equations,* October, 2009.